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A note on weighted third-order statistics for spatial point processes

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Abstract In this paper a weighted version of the directional K-statistics, that is the T function, is introduced. The weighted directional statistics is obtained by weighting points by the inverse of the conditional intensity function of the generating point process. Some theoretical results are also provided as a generalization of theorems introduced in Adelfio and Schoenberg (2009).

Keywords Third-order statistics · intensity function · point processes

Riassunto

Questo articolo presenta l'utilizzo della funzione T, nella sua versione pesata, in ambito diagnostico per processi di punto spazio-temporali. In particolare, si fa riferimento alle statistiche del terzo ordine, per descrivere gli effetti direzionali in processi complessi. Si riportano alcune proprietà teoriche.

Parole chiave *Statistiche del terzo ordine - funzione di intensità - processi di punto*

1 Introduction

Point patterns are the results of the first order and the second order characteristics effects. Both of these are similar to mean and variance of probability distribution. The first order properties are described in terms of intensity. Second order properties describe the stochastic dependency among the points that is given by the inter point interactions (Velázquez, E., et al. (2016)). The second order characteristics refer as the dispersion in the number of events occurred in per unit area for the point process.

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In point processes field the study of second-order statistics have been widely studied for analyzing attractive or repulsive characteristics of observed point patterns. One of the most popular second-order statistic is the Ripley's K-function (Ripley, 1976), which describes the expected number of further events within distance δ of any given point divided by the overall rate of the process. Estimates of K are usually provided to verify the consistence of observed patterns with the completely random Poisson point process and to identify and quantify the apparent deviation from the Poisson randomness, towards clustering or inhibition.

Though their undoubted wide field of application, first and second-order statistics do not describe the distribution of a point process completely. Indeed Baddeley and Silverman (1984) showed that differences between point patterns with identical K-function can not be identified by the use of this statistic.

When a model is fitted to a set of random points, observed in a given subset of $\mathbb{R}^d,\ d \geq 1$, diagnostic measures are necessary to assess the goodness-of-fit and to evaluate the ability of that model to describe the random point pattern behaviour. Although for temporal point processes several diagnostic tools have been already introduced, for the multidimensional case the literature is quite recent and somehow scarce. The main problem when dealing with residual analysis for point processes is to find a correct definition of residuals, since the one used in dependence models can not be used for point processes.

A widely used approach considers a stationary Poisson residual process by randomly rescaling (Meyer, 1971; Schoenberg, 1999) or thinning (Schoenberg, 2003), and investigates whether the second-order properties of the observed residuals are consistent with those of the stationary Poisson process, as in Ogata (1988). An alternative approach is to define a weighted second-order statistic, where essentially to each observed point a weight inversely proportional to the conditional intensity at that point is given. This method was adopted by Veen and Schoenberg (2005) in constructing a weighted version of the spatial K-function of Ripley (1977) (Veen, 2006).

There are often two steps involved in the diagnostic of goodness-of-fit in the theory of point processes: the transformation of data into residuals (i.e. the result of a thinning or a rescaling procedure (Schoenberg, 2003)), and the use of second-order statistics-based tests to assess the consistency of the residuals with the homogeneous Poisson process. For instance, an estimate of the autocorrelation function of residuals could indicate the amount of dependence of data which is not described by the fitted model.

However, in general, assessing the consistency of observed points with homogeneous Poisson processes often constitutes only a starting stage of a more complex analysis, necessary to build and fit a more realistic process. Considering a more general point process model rather than the stationary Poisson is often more complicated, but Zhuang (2006) considered this case for second-order residuals for various general space-time branching processes such as the Epidemic-Type Aftershock Sequence (ETAS) model.

In Adelfio and Schoenberg (2009) and Adelfio and Chiodi (2009) second-order statistics have been used to define a new diagnostic method, based on the interpretation of a weighted version of them, by introducing a martingale-based approach. A local version of these tools is proposed in Adelfio, et al. (2020), assessing the goodness-of-fit of spatio-temporal models by using local weighted second-order statistics, computed after weighting the contribution of each observed point by the inverse of the conditional intensity function that identifies the process.

For more general diagnostics, residuals defined in Adelfio and Schoenberg (2009) are here generalized to characteristics of higher order than the second one, by defining a weighted version of directional K-functions (Stoyan and Stoyan, 1994), named T-functions, that are useful to analyze anisotropy features of the generating process.

In this brief note we introduce the weighted T-function T_W obtained by weighting points by the inverse of the conditional intensity function of the generating point process assuming that it is positive and bounded away from zero in its domain. Some theoretical distributional results are also provided as just a generalization of second-order distributional results proved in Adelfio and Schoenberg (2009) looking at the weighted T-function as an extension of the K-function and, therefore, of the correlation integral for triples of points. In section 2 some definitions on point processes are provided. Some well known results about the T-function are reviewed in section 3; in section 4 the T_W statistics is introduced as well as some distributional results.

Brief conclusions are provided in section 5.

2 Point processes and second-order properties

A spatial point process is a stochastic process with realisations consisting of a finite or a countably infinite set of points in the plane. If we additionally consider the temporal occurrences for each spatially located event, we then have a spatio-temporal point process. Throughout this paper, we use N for both a spatial and a temporal point process, when no confusion arises. Point processes are introduced here by a mathematical approach that uses the definition of a counting measure on a set $X \subseteq \mathbb{R}^d, d \geq 1$, with positive values in $\mathbb{Z}_+$: for each Borel set B this $\mathbb{Z}_+$-valued random measure gives the number of events falling in B.

This section reviews some basic concepts related to point processes, reported to introduce the notation used throughout the paper. For further elaboration and references, we refer the reader to Daley and Vere-Jones (2003).

Definition 1 Point process (Cressie and Collins, 2001)
Let $(\Omega, \mathscr{A}, P)$ be a probability space and Φ a collection of locally finite counting measures on $X \subset \mathbb{R}^d$. Define $\mathscr{X}$ as the Borel σ-algebra of X and let $\mathscr{N}$ be the smallest σ-algebra on Φ, generated by sets of the form $\{\phi \in \Phi : \phi(B) = n\}$ for all $B \in \mathscr{X}$. A point process N on X is a measurable mapping of $(\Omega, \mathscr{X})$ into $(\Phi, \mathscr{N})$. A point process defined over $(\Omega, \mathscr{A}, P)$ induces a probability measure $\Pi_N(Y) = P(N \in Y), \forall Y \in \mathscr{N}$.

Given a point process N defined on the space $(X, \mathscr{F})$ and a Borel set B, the number of points $N(B)$ in B is a random variable with first moment defined by

$$\mu_N(B) = E[N(B)] = \int_{\Phi} \phi(B) \Pi_N(d\phi)$$

that is a measure on $(X, \mathscr{F})$. The measure μ_N is called the mean measure or first moment measure of N (Cressie and Collins, 2001). The second moment measure of N is given by

$$\mu_N^{(2)}(B_1 \times B_2) = E[N(B_1)N(B_k)] = \int_{\Phi} \phi(B_1)\phi(B_k)\Pi_N(d\phi),$$

with $B_1, B_2 \in \mathscr{X}$. If it is finite in $\mathscr{F}^{(2)}$ the process is said to be of second-order.

Let ds and du be small regions located at s and $u \in X$, and let $\ell(\cdot)$ be the Lebesgue measure. The first-order intensity is defined by

$$\eta(s) = \lim_{\ell(ds)\to 0} \frac{\mu_N(ds)}{\ell(ds)},$$

and the second-order intensity is given by

$$\eta_2(s,u) = \lim_{\substack{\ell(ds) \to 0 \\ \ell(du) \to 0}} \frac{\mu_N^{(2)}(ds \times du)}{\ell(ds)\ell(du)}.$$

Consider now that N is a point process on the domain $X = \mathbb{R}^2$, whose realisations are events in the form of (s). A fundamental tool is given by the conditional intensity function, defined as

$$\lambda(s|\mathscr{F}) = \lim_{ds\to 0} \frac{E[N([s,s+ds]\mathscr{F})]}{\ell(ds)}, \tag{1}$$

where $\mathscr{F}$ is the σ-algebra defined above; ds is space increment and $E[N([s,s+ds|\mathscr{F})]$ is the expected value of occurrence in the volume $\{[s,s+ds]\}$ conditioned to the filtration $\mathscr{F}$. The conditional intensity function is a function of the point history and it is itself a stochastic process depending on the past up to time t. Assuming the limit in (1) exists for each point (s,t) in the space-time domain, and that the point process is simple, then the conditional intensity uniquely characterises the finite-dimensional distributions of N (Daley and Vere-Jones, 2003). If the conditional intensity function is independent of the past history, but dependent only on the current time and the spatial locations, (1) identifies an inhomogeneous Poisson process. A constant conditional intensity characterises a stationary Poisson process.

3 Directional K-function: the T-function

Let N be a stationary planar point process with intensity λ defined on X, a subregion of $\mathbb{R}^2$ with Lebesgue measure $\ell(X)$, and $1_{\{\cdot\}}$ the Bernoulli indicator variable, x,y,z points of the state space and $\delta > 0$.

The T-function associated with N is:

$$T(\delta) = \frac{1}{2\lambda^3 \ell(A)} \mathbb{E} \sum_{x\in N\cap A} \sum_{y,z\in N} 1_{\{0<|x-y|<\delta, 0<|x-y|<\delta, 0<|y-z|<\delta\}}$$

For a homogeneous Poisson process it holds:

$$T(\delta) \equiv T_{Pois} = \frac{1}{2}\pi\left(\pi - \frac{3}{4}\sqrt{3}\right)\delta^4, \forall\delta \geq 0 \quad (2)$$

As for the estimation of K-function and correlation integral, edge effects could complicate the estimation of T.

In Schladitz and Baddeley (2000) an unbiased estimator of T is proposed, correcting for the missing δ-close triples by assigning a weight to each observed triple according to the probability of being observed. Therefore, let $A \subseteq \mathbb{R}^2$ be a compact convex window in which the process is observed, then:

$$\lambda^{\hat{3}}T(\delta) = \frac{1}{2\ell(A)} \sum_{x,y,z \in N\cap A} 1_{\{0<|x-y|<\delta, 0<|x-y|<\delta, 0<|y-z|<\delta\}} k(x,y,z), \quad (3)$$

with k a weight function, such that $\frac{1}{k(x,y,z)}$ is the measure of all transformations under which the triples (x,y,z) is inside the window. The authors suggest:

$$k(x,y,z) = \frac{\ell(A)}{\ell\left(A_x \cap A_y \cap A_z\right)}$$

such that (3) is:

$$\lambda^{\hat{3}}T(\delta) = \frac{1}{2} \sum_{x,y,z \in N\cap A} 1_{\{0<|x-y|<\delta, 0<|x-y|<\delta, 0<|y-z|<\delta\}} \frac{1_{\{\ell(A_x\cap A_y\cap A_z)\neq 0\}}}{\ell\left(A_x \cap A_y \cap A_z\right)}$$

4 Weighted T-function

We define the weighted T-function associated to a stationary point process N with intensity λ as:

$$\hat{T}_W(\delta) = \frac{1}{2\lambda_{\min}^3|A|} \sum_{x\in N\cap A} \sum_{y,z\in N} \frac{\lambda_{\min}^3}{\lambda(x)\lambda(y)\lambda(z)} 1_{\{|x-y|\leq\delta, |x-y|\leq\delta, |y-z|\leq\delta\}} \quad (4)$$

where $\lambda(s)$ is the conditional intensity function of the process with respect to some filtration $\mathscr{F}$ on A and we assume that the positive constant $\lambda_{\min} \leq \inf\{\lambda(s); s \in S\}$ exists.

Let define $D = \{x,y,z : 0 \leq |x-y| \leq \delta, 0 \leq |x-z| \leq \delta, 0 \leq |y-z| \leq \delta,\}$; since for any function f on A, $\int_A f dN$ can be written as the sum $\sum_{i:x_i\in A} f(x_i)$, if edge effects are ignored and if the boundary of A are assumed to be sufficiently regular, the following theorem holds.

Theorem 1 *Let N be a point process defined on $A \subseteq \mathbb{R}^2$ with conditional intensity function $\lambda(\cdot|\mathscr{F})$, defined with respect to some filtration $\mathscr{F}$ on A, positive and bounded away from zero and $D = \{x,y,z : 0 \leq |x-y| \leq \delta, 0 \leq |x-z| \leq \delta, 0 \leq |y-z| \leq \delta,\}$. Then the expected value of $\hat{T}_w(\cdot)$ defined in (4) is T_{pois} (see eq. (2)).*

Proof

$$E[\hat{T}_W(\delta)] =$$

$$= E\left[\frac{1}{2\lambda^3_{\min}|A|}\sum_{x,y,z\in N\cap A} 1_{\{0<|x-y|<\delta,0<|x-y|<\delta,0<|y-z|<\delta\}}\frac{\lambda^3_{\min}}{\lambda(x)\lambda(y)\lambda(z)}\right]$$

$$= E\left[\frac{1}{2|A|}\int_D \frac{1}{\lambda(x)\lambda(y)\lambda(z)} dN(x)dN(y)dN(z)\right]$$

taking conditional expectations on $\mathscr{F}_x, \mathscr{F}_y$ and on $\mathscr{F}_z$

$$= E\left[\frac{1}{2|A|}\int_D \frac{1}{\lambda(x)\lambda(y)\lambda(z)} \lambda(x)\lambda(y)\lambda(z) d\ell(x)d\ell(y)d\ell(z)\right] \tag{5}$$

$$= E\left[\frac{1}{2|A|}\int_D d\ell(x)d\ell(y)d\ell(z)\right]$$

from Schladitz and Baddeley (2000), Prop. 1, and Slinyak's theorem (Stoyan et al., 1995):

$$= E\left[\frac{1}{2|A|}|A|\int_{b(0,\delta)}\int_{b(0,\delta)} 1_{\{0<|x-y|<\delta\}} d\ell(x)d\ell(y)\right]$$

$$= T_{Pois}$$

where $b(0,\delta)$ is the closed ball of radius δ centered at 0 and T_{Pois} is defined in (2).

By a martingale approach it is possible to show that $\hat{T}_W(\delta)$ is asymptotically normally distributed. First let consider the weighted T-function for a time point process N with realizations $t_1, t_2, \dots, t_n$ on $[0,T] \in \mathbb{R}$ with Lebesgue measure T, defined by:

$$\hat{T}_W(\delta) \approx \frac{1}{(\lambda^3_{\min}T)}\sum_i^n \omega_i \sum_{j\neq i}^n \omega_j \sum_{k\neq i,j}^n \omega_k 1_{\{0<|t_i-t_j|<\delta,0<|t_i-t_k|<\delta,0<|t_j-t_k|<\delta\}}, \tag{6}$$

with $\omega_l = \frac{\lambda_{\min}}{\lambda(t_l)}, \forall l$ and $\lambda(t)$ the conditional intensity function of the process with respect to some filtration $\mathscr{F}_t$ on $[0,T]$. Let ζ_1 be the distance between two points of the pairs $\{t_p,t_q\},\{t_p,t_r\},\{t_q,t_r\}_{p\neq q\neq r} \in [0,T]$ and

$$I^w_{pqr}(\delta)|_T = 1_{\{0<|p-q|<\delta,0<|p-r|<\delta,0<|q-r|<\delta,T\}}\frac{\lambda^3_{\min}}{\lambda(p)\lambda(q)\lambda(r)},$$

where $1_{\{0<|p-q|<\delta,0<|p-r|<\delta,0<|q-r|<\delta,T\}}$ is the indicator function such that:

$$1_{\{0<|p-q|<\delta,0<|p-r|<\delta,0<|q-r|<\delta,T\}} = \begin{cases} 1, & \text{if } 0 < \zeta_1 < \delta, \\ 0, & \text{otherwise} \end{cases}$$

Therefore the following theorem holds.

Theorem 2 *Let N be a temporal point process defined on $\mathbb{R}$ with conditional intensity function $\lambda(t|\mathscr{H})$ positive and bounded away from zero. The expected value of $\hat{T}_W(\delta)$ weighted by the inverse of the conditional intensity function does not depend on its weights.*

Proof

$$\begin{aligned}
&E\left[\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} I^w_{pqr}(\delta)|_T dN(p)dN(q)dN(r)\right] \\
&= E\left[\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} 1_{\{0<|p-q|<\delta,0<|p-r|<\delta,0<|q-r|<\delta,T\}} \frac{\lambda^3_{\text{min}}}{\lambda(p)\lambda(q)\lambda(r)} dN(p)dN(q)dN(r)\right] \\
&\textit{taking conditional expectations on } \mathscr{F}_p, \mathscr{F}_q \textit{ and on } \mathscr{F}_r \\
&= E\left[\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} 1_{\{0<|p-q|<\delta,0<|p-r|<\delta,0<|q-r|<\delta,T\}} \frac{\lambda^3_{\text{min}}}{\lambda(p)\lambda(q)\lambda(r)} \lambda(p)\lambda(q)\lambda(r) d\ell(p)d\ell(q)d\ell(r)\right] \\
&= E\left[\lambda^3_{\text{min}} \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} 1_{\{0<|p-q|<\delta,0<|p-r|<\delta,0<|q-r|<\delta,T\}} d\ell(p)d\ell(q)d\ell(r)\right] \\
&= \lambda^3_{\text{min}} T\delta^3
\end{aligned}$$

Let $I^w_{pqr}(\delta)|_{t_i}$ and $I^w_{pqr}(\delta)|_{t_i - t_{i-1}}$ be defined as $I^w_{pqr}(\delta)|_T$, conditioning on the points that occur before t_i and in the interval $[t_{i-1}, t_i]$, for any $i = 1, 2, \ldots, n$, respectively.

To prove the martingale characterization of the weighted T function (6) let

$$\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} I^w_{pqr}(\delta)|_{\tau_i} dN(p)dN(q)dN(r)$$

be the number of triples of points with elements occurring both up to τ_i, for any i, where τ_i is the last point less than or equal to t_i such that no points are in $(\tau_i, \tau_i + \delta)$. Formally $\tau_i = \sup\{\tau : N(\tau) = 1, N(\tau, \tau + \delta) = 0, \tau + \delta < t_i\}$ and $\tau_i = 0$ if no such τ exists. To simplify τ_i can be considered as the left end-point of the last gap prior to t_i of size at least δ.

Then, let $Z(t_i) = \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} I^w_{pqr}(\delta)|_{\tau_i} dN(p)dN(q)dN(r) - \lambda^3_{\text{min}}\delta^3\tau_i$ for any i. Assuming that, conditionally on $Z(t_{i-1})$, such kind of gaps exists in correspondence of each t_i, then the knowledge about the past up to any point t_{i-1} does not give any information about t_i; indeed no pairs within distance δ crosses t_{i-1} and for each i, τ_i is an $\mathscr{H}_{t_i}$-stopping time and thus $Z(t_i)$ is measurable (Daley and Vere-Jones, 2003).

The martingale characterization of $T_W(\delta)$ is therefore proved by the application of Theorem 5 in Adelfio and Schoenberg (2009) and the asymptotic normal distribution is then derived by the appropriate generalizations of Theorem 6.

Therefore, considering a spatial domain, a filtration $\mathscr{F}$ on $A \in \mathbb{R}^2_+$ is defined on the basis of a specified ordering, such that given two points $s' = (x', y')$ and $s'' = (x'', y'') \in \mathbb{R}^2_+$, say $s' \leq s''$ if the Euclidean distance from s' to the origin (s_0) is less than the Euclidean distance from s'' to the origin, i.e $s' \leq s'' \Leftrightarrow |\overline{s's_0}| \leq |\overline{s''s_0}|$.

The filtration $\mathscr{F}(s)$ on the complete probability space $(\Omega, \mathscr{A}, P)$, is defined as the increasing family of sub-σ-algebras of $\mathscr{A}$ such that if $s' \leq s''$ then $\mathscr{F}(s') \subseteq \mathscr{F}(s'')$, assuming that the point process N vanishes on the axes (for details see Adelfio and Schoenberg (2009)).

Therefore, let ζ_2 be the distance defined in terms of the defined order on A, between the three pairs $\{s_p, s_q\}, \{s_p, s_r\}, \{s_q, s_r\}_{p \neq q \neq r} \in A$ and

$$I^w_{pqr}(\delta)|_A = \frac{1_{\{0<|p-q|<\delta, 0<|p-r|<\delta, 0<|q-r|<\delta, A\}}}{\lambda(p)\lambda(q)\lambda(r)}$$

where $1_{\{0<|p-q|<\delta, 0<|p-r|<\delta, 0<|q-r|<\delta, A\}}$ is the indicator function such that:

$$1_{\{0<|p-q|<\delta, 0<|p-r|<\delta, 0<|q-r|<\delta, A\}} = \begin{cases} 1, & \text{if } 0 < \zeta_2 < \delta, \\ 0, & \text{otherwise} \end{cases}$$

Define $I^w_{pqr}(\delta)|_{A_h}$ as $I^w_{pq}(\delta)|_A$ just conditioning on the points that occur in the space area A_h.

The introduced ordering, and assuming N is simple, make possible to move from the $\mathbb{R}^2_+$ to $\mathbb{R}$; as a consequence the results provided for time processes can be extended to spatial ones.

5 Conclusion

In this paper the weighted T-function is introduced as a generalization of weighted second-order statistics introduced in Adelfio and Schoenberg (2009). Although some theoretical results of T seem to be unobtainable for most point process models apart from the homogeneous Poisson process, we introduce some theoretical results about its weighted version. These results could be used for diagnostic analysis and applications when anisotropy features of the generating process are investigated, with the aim to discriminate between different types of point processes even if second-order of these processes coincide.Indeed, the advantage of these approaches relies on the fact that, alternatively to the more classical diagnostic methods based on transformation of the data into residuals as a result of a thinning or a rescaling procedure, the proposed weighted measures directly apply to data without assuming homogeneity nor transforming the data into residuals, eliminating thus the sampling variability due to the use of a transforming procedure, independently of any particular model assumption on the data, that is for whatever is the generator model of the process.

References

Adelfio, G., Chiodi, M., 2009. Second-order diagnostics for space-time point processes with application to seismic events. Environmetrics 20, 895 - 911

Adelfio, G., Schoenberg, F. P., 2009. Point process diagnostics based on weighted second-order statistics and their asymptotic properties. Annals of the Institute of Statistical Mathematics 61 (4), 929–948.

Adelfio, G., Siino, M., Mateu, J. et al. 2020. Some properties of local weighted second-order statistics for spatio-temporal point processes. Stoch Environ Res Risk Assess 34, 149 - 168.

Baddeley, A. J., Silverman, B. W., 1984. A cautionary example on the use of second-order methods for analyzing point patterns. Biometrics 40, 1089–1093.

Cressie, N. , Collins, L. B., 2001. Analysis of spatial point patterns using bundles of product density lisa functions. Journal of Agricultural, Biological, and Environmental Statistics 6, 118–135.

Daley, D. J., Vere-Jones, D., 2003. An introduction to the theory of point processes, 2nd Edition. New York: Springer-Verlag.

Meyer, P., 1971. Dèmonstration simplifée d'un théorème de knight Lecture Notes in Mathematics 191, 191–195.

Ogata, Y., 1988. Statistical models for earthquake occurrences and residual analysis for point processes, Journal of the American Statistical Association 83, 9–27.

Ripley, B. D., 1976. The second-order analysis of stationary point processes. Journal of Applied Probability 13 (2), 255–266.

Ripley, B., 1977. Modelling spatial patterns (with discussion) Journal of the Royal Statistical Society, Series B 39, 172–212.

Schladitz, K., Baddeley, A. J., 2000. A third order point process characteristic. Scandinavian Journal of Statistics 27, 657–671.

Schoenberg, F. P., Stochastic Processes and their Applications 81 (1999) 155–164.

Schoenberg, F. P. , 2003. Multi-dimensional residual analysis of point process models for 420 earthquake occurrences Journal American Statistical Association 98, 789–795.

Stoyan, D., Kendall, W. S., Mecke, J., 1995. Stochastic geometry and its applications. Wiley, Chichester.

Stoyan, D., Stoyan, H., 1994. Fractals, random shapes and point fields. Wiley, Chichester.

Veen, A.,2006. Some Methods of Assessing and Estimating Point Processes Models for Earthquake Occurrences, Ph.D. thesis, UCLA.

Veen, A., F. P. Schoenberg, 2015. Assessing spatial point process models using weighted k-functions: Analysis of california earthquakes UC's eScholarship Repository.

Velázquez, E., Martínez, I., Getzin, S., Moloney, K. A., & Wiegand, T. 2016. An evaluation of the state of spatial point pattern analysis in ecology. In Ecography. https://doi.org/10.1111/ecog.01579

Zhuang, J. , 2006. Second-order residual anaysis of spatio-timporal point processes and applications in model evaluation Journal of the Royal Statistical Society, Series B 68, 635–653.